The Conic Sections

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This description is intended for students of analytic geometry, and is intended first to introduce the subject, and then to apply the mathematics to a few practical problems. In this way, one can see why this topic drew so much attention from mathematicians throughout the ages. The student is also referred to an excellent on-line exposition of conic sections (Erbas and Kim).

I. History

While this study appears to have begun with Menaechmus, a Thracian and friend of Plato, it was Apollonius of Perga who studied these geometric curves in depth, and wrote extensively about them. The methods of these early geometers seem clumsy today, and we owe our modern insights to Rene Descartes. Generating the curves from a cone was perhaps understood by Menaechmus and Euclid, but what survives are the books of Apollonius, and the acknowledgement that he was indeed the master of the conic sections.

The perils of a book are many; much of the work of ancient world mathematicians has been lost, partly due to neglect, but also through wars, pogroms and other persecutions, and of course the physical decay of the manuscripts. Fortunately, the Arabs took an interest in Greek mathematics and preserved much of Apollonius' work in Arabic; later Renaissance scholars obtained these works and of course carried the mathematics further.

Some concepts I thought originated with the ancients didn't. Apollonius apparently never made use of a directrix; Pappus of Alexandria did, in his discussion of the work of Euclid. More recent mathematicians have carried this study to a simpler form, and we have benefited from engineering developments where these interesting geometric curves play a part. I will try to present some of these applications here.

II. Deriving the conic sections from a right circular cone

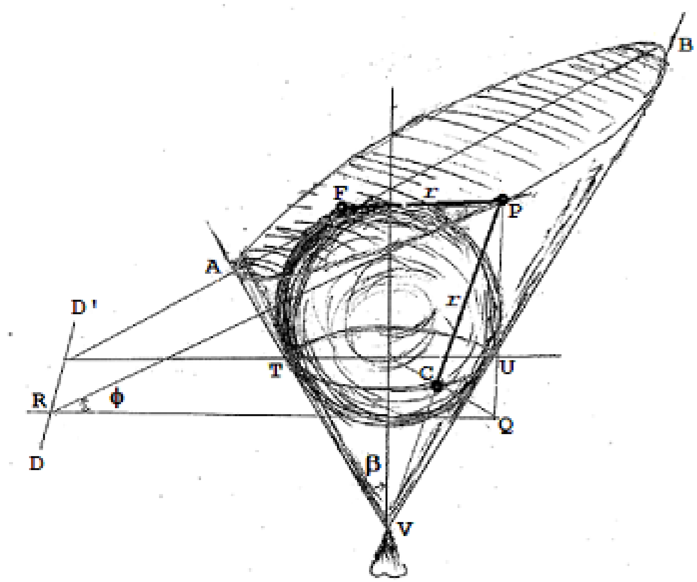

Fig. 1.

A right circular cone is one with circular crossection and axis perpendicular to its circular base. Fig. 1 shows such a cone, stood on its end, which has been cut by a plane ABP at an angle ϕ to the horizontal. There is a hint that the cone may be extended past the vertex to a second cone opening up downward. We are interested in the shape of the curve defined by the intersection of this plane and the cone's surface, our so-called *conic section.* A sphere has been imbedded in the cone, tangent to its surface, and also tangent to the cut at a single point F in the figure, which we'll call the *focus.* This sphere is an invention of Dandelin in the 19th century. Danby has a nice exposition of the construction in an appendix to his *Fundamentals of Celestial Mechanics*, and I have followed his convention here. In the case where the cut is steep enough and cuts the lower cone, another sphere may be constructed in the lower cone tangent to the other focus. (A parabola will cut parallel to the far side of the upper cone, and will miss the lower cone; it has only one focus.) For all these cases, Dandelin's construction may be used, and indicates the location of the focus (or foci). (We shall ignore certain other cuts through the cone that form degenerate lines or points.)

Consider a point P which lies on the curve defined by the cut with the cone. A line PF will be tangent to the constructed sphere at F, and likewise, if a line is drawn from P toward the cone's vertex V, but stopping at C where it is tangent to the sphere, then we must have PF = PC, both shown here to have length r. (All lines from a point to tangents to the sphere will be equal in length.)

Consider also the line D-D', where the plane defined by the cut ABP and the plane TUC defined by the circle where the sphere is imbedded, meet. This is called the *directrix*. The two planes cross with an angle ϕ. Now construct a line from P to the TUC plane, normal to that plane and striking it at Q. The distance PC = PQ sec β, where β is the angle the cone's surface makes with its axis at the vertex V. PQ is of course parallel with the cone's axis. Let l be the length of a line from P to R on the directrix, and perpendicular to the directrix. PQ can be seen from the triangle PQR to have length PR $\sin\phi$. We now have an important result:

$r = l$ sec β sin ϕ.

That is, for every point P on the curve defined by the intersection of the oblique plane ABP, that is every point P on our conic section, lies a distance r from its focus, and a distance $l = r/e$ from its directrix, where e is a constant we'll call the eccentricity of the conic section. As one can see, a circle has no finite directrix.

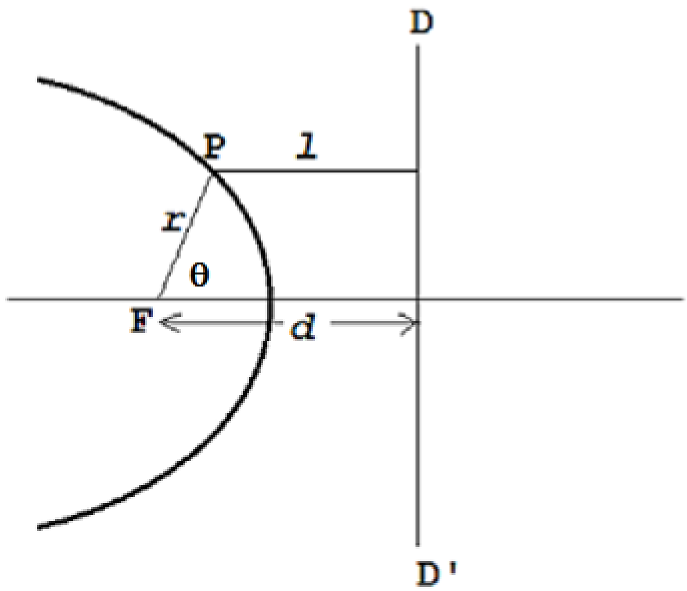

Fig. 2.

Now we can construct any conic section (except a circle) by drawing a point for its focus, and a line for its directrix, and then lay out the curve. We find:

$r = e(d - r\ \cos\ \theta)$,or

$r = p/(1 + e\cos\ \theta)$,

where we have defined a paremeter $p = ed$. For the ellipse, we have $0 < e < 1$. Let us put this into Cartesian coordinates, noting that
$x = r\ \cos\ \theta$,we have

$\sqrt{x^2 + y^2} = p - ex$,

$x^2 + y^2 = p^2 - 2pex + e^2x^2$,

$x^2(1 - e^2) + 2pex + y^2 = p^2$,

$x^2 + 2pex/(1-e^2) + y^2/(1-e^2) = p^2$.
Completing the square,

$\left[x + pe/(1-e^2)\right]^2 + y^2/(1-e^2) = p^2/(1-e^2) + p^2e^2/(1-e^2)^2 = p^2/(1-e^2)^2$.
Let $a = p/(1-e^2)$,the semi-major axis, so then

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1,$$

and we note that when $x + ae = 0$,$y^2 = a^2(1-e^2)$,which we'll call b^2, the square of the semi-minor axis. We recognize that we have an example of the standard formula for an ellipse in rectangular coordinates, which is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

for the ellipse with center at coordinates (h,k). The displacement ae we now see is the position of one of the foci (on the right).

The parabola is the case when $e = 1$. For this case, set $p = 2q$,q the distance from focus to the curve's vertex, where the curve intersects the axis of symmetry. This makes $d = 2q$.We then have

$\sqrt{x^2 + y^2} = 2q - x$,
$y^2 = -4qx + 4q^2$,or

$$y^2 = 4q(q-x).$$

which we see as a curve opeining up to the left, with an x domain of $(-\infty, q]$.
For the hyperbola, we set the semi-major axis $a = p/(e^2-1)$,whence

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

where this time we set the semi-minor axis $b = a\sqrt{e^2-1}$.

III. Centered Formulae

A standard form for writing these equations depicts the curves as having centers located at $x = h$ and $y = k$, with the ellipse and two-lobed hyperbola being centered at coordinate(h,k). Often, additional formulae are given with the long axis running vertically as opposed to what we have here, which assumes the x axis to be the long axis. Since we can easily get the other set by swapping x and y in our formulae, there is no reason to present the other orientation here.

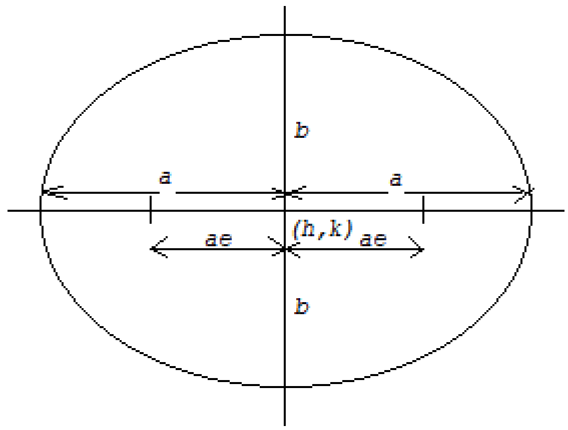

Fig. 3.

Ellipse

See Fig. 3. The equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

where a is called the semi-major axis, and b the semi-minor axis. The eccentricity $e = \sqrt{a^2 - b^2}/a$. Each directrix lies a distance a/e from the center, at coordinates (h, k). There are two focal points, located a distance ae on either side of the center, and lying on the semi-major axis.

If we center the ellipse at the origin, the polar version of the equation can be written:

$$r^2 = \frac{a^2(1-e^2)}{1 - e^2 \cos^2 \theta}.$$

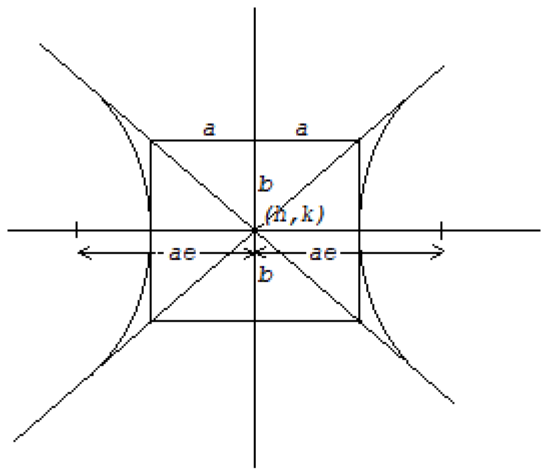

Fig. 4.

Hyperbola

See Fig. 4. The equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

where again we refer to a and b as semi-major and semi-minor axes, but note that the curve lies outside the box thus defined, being tangent to the vertical sides at the axis of symmetry. As before, the figure is centered at (h, k). For large x, (or large negative x), the curve draws closer and closer to the diagonal lines running through the corners of the box – these are called *asymptotes*. The eccentricity is $e = \sqrt{a^2 + b^2}/a$. Again, the foci are distant ae from the center; since $e > 1$ that means they lie outside the box shown. The directrixes are distant a/e from the center.

The polar equation for the hyperbola centered at the origin is

$$r^2 = \frac{a^2(e^2-1)}{e^2\cos^2\theta - 1}.$$

The above Cartesian coordinate formulae become rather complicated when the coordinate system is rotated. Any quadratic equation in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

is a conic. Consider the *discriminant*, $\Delta = B^2 - 4AC$. If $B = 0$, and $A = C$, the equation represents a circle. Otherwise if $\Delta < 0$, it is an ellipse.If $\Delta = 0$, it is a parabola; if $\Delta > 0$,it is a hyperbola. (We are ignoring some degenerate cases, such as $A = B = C = 0$,etc.) To see the effect of coordinate rotation, recalling the equations for rotation (in two dimensions):

$$x' = x\cos\theta + y\sin\theta,$$

$$y' = -x\sin\theta + y\cos\theta,$$

we ask what is the angle of rotation θ which will transform the above polynomial into, say, the equation of an ellipse in standard form? Let us start with

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

and it has been rotated to this form via the transformation equations for x' and y'. Writing $b^2 = a^2(1-e^2)$, we have

$$a^2(1-e^2)(x\cos\theta + y\sin\theta)^2 + a^2(-x\sin\theta + y\cos\theta)^2 = a^4(1-e^2),$$

where we have multiplied through by the common denominator. a^2can now be eliminated from both sides, and we wind up with

$$\begin{gathered}(\cos^2\theta - e^2\cos^2\theta + \sin^2\theta)x^2 \\ +(2\sin\theta\cos\theta - 2e^2\sin\theta\cos\theta - 2\sin\theta\cos\theta)xy \\ +(\sin^2\theta - e^2\sin^2\theta + \cos^2\theta)y^2 \\ = a^2(1-e^2).\end{gathered}$$

Note we've left out the terms $Dx + Ey$. These only effect a translation of coordinate, getting us the h, k offsets seen in the centered formula; while these will get rotated into new values, they can be safely neglected. Notice, with some obvious simplification, we now have:

$$A = 1 - e^2\cos^2\theta,$$

$$B = -2e^2\sin\theta\cos\theta,$$

$$C = 1 - e^2\sin^2\theta.$$

We can neatly tie this up:

$$\frac{A-C}{B} = \frac{-e^2(\cos^2\theta - \sin^2\theta)}{-2e^2\sin\theta\cos\theta} = \cot 2\theta.$$

So it is we can always transform the general polynomial into one of the centered conic equations. (The argument for transforming the hyperbola proceeds just as it did with the ellipse, only we have a common denominator of $a^4(e^2-1)$ and the y'^2 term is negative.)

IV. Vertex Formulae

A conic in general may be written in a form with the long axis horizontal (also the axis of symmetry), and one of its vertices located at the origin. The vertex as defined here is the point where the curve crosses that axis of symmetry, which we'll take to be the x axis. This vertex formulation is attributed to Schwarzschild (see Conic Constant Wiki page in references). Any conic curve may be written

$$(K + 1)x^2 - 2Rx + y^2 = 0,$$

where K is the Schwarzschild constant, or *conic constant*, or "K constant" for the curve, where $K = -e^2$, e the eccentricity, and R is the instantaneous radius of curvature at the vertex.

This form has the advantage of handling oblate ellipses without any fuss with imaginary numbers. Surfaces are generated when these curves are rotated about the x axis, and the application of these to surfaces of optical elements is the motivation. The various curves we obtain are:

conic const.	curve
$K < -1$	hyperbola or hyperboloid
$K = -1$	parabola or paraboloid
$-1 < K < 0$	prolate ellipse or ellipsoid, like a Rugby ball
$K = 0$	circle or sphere
$K > 0$	oblate ellipse or ellipsoid, like the planet Jupiter

These curves are symmetric about the x axis, and we could conveniently substitute r^2 for y^2 with $r^2 = y^2 + z^2$, and x is along the symmetry axis, also the optical axis in an optical system. This formulation is quite useful in problems of optical design. It is conventional for a prescription for an optical design to specify locations of the vertices of the various curves, and to give their radii and also the K constants for surfaces of the optics. Of course, optical surfaces can be more complex than this, but conics are special cases that are significant for their optical properties. We'll explore this a little in this section.

Let's give a little attention to the idea of instantaneous curvature at the vertex. For the circle (or sphere) the radius is of course the same everywhere, and, putting the circle into the Schwarzschild form with the center in the $+x$ direction, $(x - R)^2 + y^2 = R^2$,becomes

$$x^2 - 2Rx + y^2 = 0.$$

Here, $K = 0$, as mentioned above. Solving for x, given a y, this gives (taking the – root to put the center on the right),

$$x = R - R(1 - y^2/R^2)^{1/2},$$

applying the binomial theorem, we can write x as a series

$$x = R - R\left[1 - \frac{y^2}{2R^2} + \frac{y^4}{8R^4} - \cdots\right],$$

or, narrowing our attention to the immediate vicinity of the vertex, we make the approximation

$$x \approx \frac{y^2}{2R},$$

given y sufficiently small relative to the radius. There is of course a more direct way to calculate the surface radius using calculus, but the above approach is more intuitive. Anyone who has worked in an optical shop will recognize this as the "spherometer formula," telling us for a given R how much "sag" there is in the surface, or from reading the sag with a dial gauge, we can compute the radius of curvature. Let's now look at an elliptical surface of the prolate (football) type with vertex on its left lobe:

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$b^2x^2 - 2b^2ax + a^2y^2 = 0,$$

taking the $-$solution of the quadratic equation,

$$x = a - a(1 + y^2/b^2)^{1/2},$$

$$x = a - a\left[1 + \frac{y^2}{2b^2} - \frac{y^4}{8b^4} + \cdots\right],$$

approximating, as with the circle,

$$x \approx \frac{ay^2}{2b^2},$$

so the radius of curvature is

$$R = b^2/a.$$

Now, noting that $b^2 = (1-e^2)a^2$, in our equation just solved, we can put the ellipse into Schwarzschild form:

$$(1-e^2)x^2 - 2b^2x/a + y^2 = 0.$$

If we repeat this process for the hyperbola, we obtain the same result for the radius of curvature, and get

$$(1-e^2)x^2 - 2b^2x/a + y^2 = 0,$$

remembering for the hyperbola, $e^2 > 1$ and $(1-e^2) < 0$,in keeping with $K < -1$.

V. Optical Properties of the Conic Sections

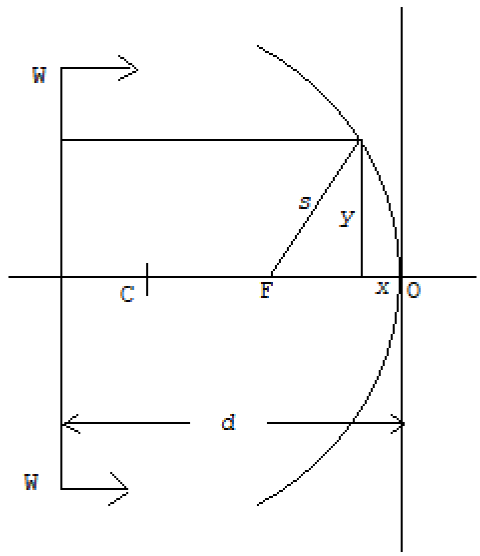

Fig. 5.

The Parabola

Consider a plane wavefront W-W moving to the right in Fig. 5. Let's freeze it in time at some position a distance d from the position of a reflective parabola whose vertex lies at O. If this wave is to converge to a focus at F, then we need to have the same path distance along every ray drawn from W-W to the mirror, both as it travels through space, and then after it is reflected. Why a plane wave? That is what every light wave would look like after it arrives from a great distance from its origin ("object point"). Rays, once really thought to exist as such, describe lines normal to the wavefront, and we find them useful in making optical calculations. Consider two rays. One is drawn from W-W through the center of curvature C (for a narrow zone near the vertex) and then to the vertex O. This ray is reflected, and after reaching F, it has traveled a distance $d+f$, where$f$ is the focal length of the parabola (distance of focus from the vertex).

Now the equation for the parabola is

$$y^2 = 4fx.$$

Let's compute the path distance along a ray that hits the parabola a distance y above the axis, reflects off the mirror surface, and then passes through point F. That would be: $P.D. = d - x + s = d - x + \sqrt{(f-x)^2 + y^2} = d - x + \sqrt{f^2 - 2fx + x^2 + 4fx} = d - x + (f + x) = d + f$.

So we see that the path distance from wavefront to focus is the same for any ray perpendicular to the wavefront W-W. What really happens now when we now hit the forward button and allow the wave to strike the mirror, first the outer edges, then later the more central parts will be reflected so as to make a spherical surface that converges toward F.

Side note: According to ray optics, there would then be an infinitely bright spot at F, owing to the excellent properties of the mirror. Mother Nature intervenes, however. She causes portions of the incoming sphere to interfere with one another, so as to create, not a perfect spot, but a blob of light surrounded by concentric rings, the “diffraction rings.” The larger the mirror, the smaller the blob (and brighter), but it remains a blob.

Now, equal path lengths through an optical surface or optical system are usually what is sought in designing lens and mirror systems. This means each point on the object is brought (hopefully) to a single point in the image near the focus. Of course, this never happens in practice; what happens instead is that the imaging is imperfect; the system is then said to have “aberrations,” and the minimization of these is the goal of the optical designer. The parabola, we just saw, makes a sharp image of a distant object point, but what we didn’t show is the blur that happens when the object lies a little ways off the central axis. It resembles a comet, or maybe somebody’s hair-do, and the aberration seen in the parabolic mirror is known as “coma.” But a spherical mirror would be worse-- it wouldn’t even focus the object point on the axis, and that resulting blur is known as “spherical aberration.” A spherical mirror can only image an object at its center.

The Ellipse

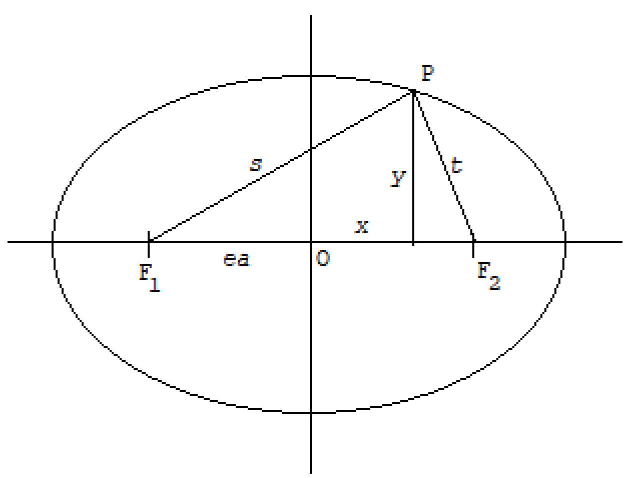

Fig. 6.

The ellipse has two foci, and you might imagine that if an object were placed at one of these, and the surface were reflective, its image at the other focus would be perfect. You would be right, and we’ll demonstrate that here. Let P be any point on the surface, and let us compute the entire path distance from F1, at a focal point, to P, and then to F2, the other focus. The total path distance is $s + t = \sqrt{(ea + x)^2 + y^2} + \sqrt{(ea - x)^2 + y^2}$. Recall the foci are spaced a distance ea from center. Looking at the s segment, $s^2 = (ea + x)^2 + y^2 = e^2a^2 + 2eax + x^2 + (1 - x^2/a^2)b^2$,and noting that $b^2 = (1 - e^2)a^2$,we conclude that $s^2 = (a + ex)^2$. Similarly, $t^2 = (a - ex)^2$. In this sort of calculation, we must be careful the path length is always positive. Since ex can never exceed a in length, it is necessary to take the + square root of both s^2 and t^2, getting (no surprise),

$$s + t = 2a.$$

Since the choice of P is arbitrary, we see that the path distance between foci is constant, so that one focus will be imaged at the other.

This property is useful for drawing an ellipse: We take a loop of wire of length 2a(1 + e), place two pins at the foci, and stretch the wire over the two pins and a pencil, and draw the curve. This trick is used in a woodworking shop whenever a table with an elliptical top is needed – two nails are driven into the stock to be used as a template, and the curve laid out, and cut with a jig saw. We then use the template with a router to generate the table top itself.

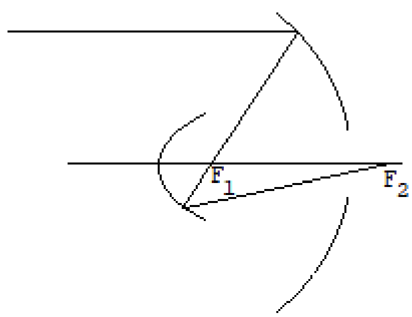

Fig. 7.

An optical design, the Gregorian reflecting telescope, is constructed with an elliptical mirror for its secondary, as seen in Fig. 7. A parabolic primary mirror with focus at F_1, coinciding with one of the foci of a smaller elliptical mirror, operates as a so-called “compound telescope” and produces an image at F_2, where it can be viewed with an eyepiece. The primary mirror is perforated. Note that the secondary mirror presents an obstruction, which slightly degrades the image (diffraction effects). The ratio of the distance f_2 from the ellipsoid’s vertex to F2 divided by $f_1$the distance to F1, is $m = f_2/f_1$serves to magnify the image formed at F1. It is called the “amplification ratio” of the secondary, and causes the telescope to have its primary mirror focal length thus extended. A long focus system is thus accomodated in a shorter space, an advantage as telescopes become larger.

The Hyperbola

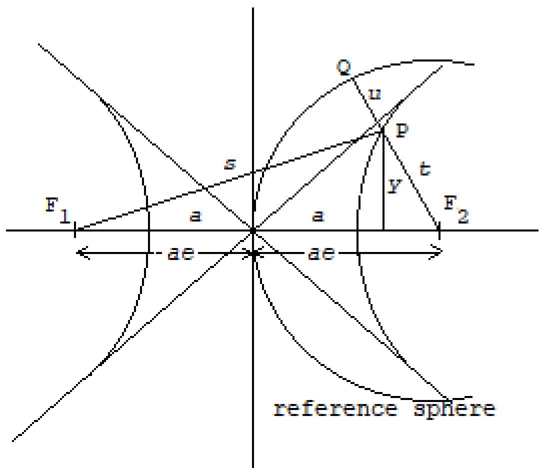

Fig. 8.

Let a ray pass from the left focus F_1 to the right lobe of the hyperbola, intersecting it at P. It then is reflected at P and strikes a sphere at Q. This *reference sphere*, as we'll call it, is centered on F_2, the right focus. Let the sphere have radius ae. Since the ray cannot strike F_2, that focus is said to be *virtual.* However, we'll show that all rays from F_1 will experience the same path distance between there and where they strike the reference sphere, and that total path distance is $2a + ae$. We can write the equation of the hyperbola as

$$y^2 = (e^2 - 1)(x^2 - a^2),$$

so the path distance from F1 to Q can be found from

$$s^2 = (ae + x)^2 + (e^2 - 1)(x^2 - a^2),$$

$$s = a + ex.$$

The remaining distance to the reference sphere is $u = ae–t$, where

$$t^2 = (ae - x)^2 + y^2 = (a - ex)^2.$$

Now we need to be careful in extracting the square root here, since $e > 1$ and $x \geqq a$. The resulting distance must be positive, so we take $t = ex–a$. The total path distance then is $2a + ae$,as was to be demonstrated. Usually this result is presented as simply $s - t = 2a$. Our use of a reference sphere avoided making any of the paths negative.

The rules of ray optics say that one has equal angles with a normal vector to the surface which a ray makes on reflection; the angle of incidence equals the angle of reflection. It is possible to carry out these calculations for the conics using vectors to accomplish the above demonstrations, but the author considers the path distance argument to be easier to carry out.

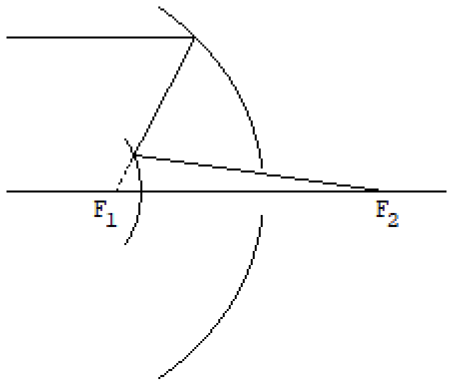

Fig. 9.

A practical use of a hyperbolic reflector is found in the Cassegrain design (see Fig. 10). A parabolic primary mirror sends the light to what would be a focus at F_1, but the rays are intercepted by a hyperbolic secondary mirror whose first focus coincides with F_1. The other focus of the hyperbola is at F_2, where the final image is formed. F_1 is said to be virtual. The parabolic mirror is perforated so that the light may pass through. Like the Gregorian design, the central obstruction prevents a truly diffraction-limited image to be formed, but if the obstruction is kept small (secondary mirror close to F_1), then the image is quite good. The Cassegrain has a smaller obstruction than an equivalent Gregorian, and requires a shorter tube for mounting the optics; hence it is much more commonly found in astronomical telescopes. Again, the amplification ratio, $m = f_2/f_1$, applies to this design.

I spoke earlier about a sphere being unable to image a faraway point, resulting in "spherical aberration." It is as true of lenses as with mirrors. However, it is possible to correct for this problem in multiple lens systems, with some optics producing aberrations of the opposite sign to others. For a single lens it remains true unless the surface is made non-spherical (an "aspherical surface"). Let's design such a lens.

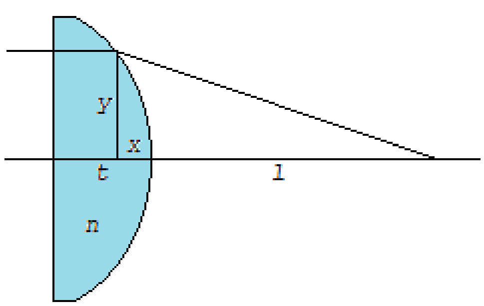

Fig. 10.

When light enters glass, or similar transparent material (a "dielectric" material), it is slowed down by a factor n, the refractive index of the material. This shortens the wavelength of the light (the separation of wave crests), and so our notion of path distance, used to make sure all rays arrive at their destination via the same length, needs to be modified. As you might surmise, if the waves arrive at their destination out of phase, a blur will result; not a sharp focus. Hence the path distance through the center of our lens will not be t, its thickness, but nt instead. For any real material, $n > 1$. At optical wavelengths, n ranges from about 1.45 to 2.05 (Schott catalogue), with common "crown glass" being about 1.5.

Assume a lens, plane on its front side, and curved on its back, and we will try to learn what shape that curve must have for equal path distances to the focus. Light arrives from a distant source in a plane wave parallel to the front surface. Counting from that front surface, the central ray then covers an optical path distance $nt + l$,where l is the distance from the lens to the focus. Now, the path distance for a ray at height y will be $n(t + x) + \sqrt{(l - x)^2 + y^2}$, which we require to be the same, namely, $n(t + x) + \sqrt{(l - x)^2 + y^2} = nt + l$.

Note that x should be considered negative if we are to follow sign conventions most commonly used, since the curve bends to the left. Putting the radical to one side, we have on squaring, $(l - x)^2 + y^2 = (l - nx)^2$,
and we see there is no dependence on the thickness, t. Finally, we obtain

$$(1 - n^2)x^2 + 2l(n - 1)x + y^2 = 0,$$

which we recognize as a conic in the Schwarzschild format, and we see that it is a hyperbola with $e = n, K = -n^2$, and $e > 1$. The radius of curvature is -$(n - 1)l$ at the vertex.

One further optical application of conic surfaces may be found in the attempts by Ermanno Bora, Laval University, Canada, to produce an astronomical objective mirror by spinning a dish full of mercury. He has partnered with Paul Hickson, University of British Columbia, Canada in constructing several such zenith telescopes, and one known as the International Liquid Mirror Telescope (ILMT) is being constructed in northern India. While restricted to look only straight up, these telescopes can scan in depth a narrow path across the sky, and can serve useful survey purposes.

To understand the operation of a liquid mirror, consider the forces on an element of liquid: A constant downward force of mg,where m is the mass of a small volume of the liquid, and g the acceleration of a particle due to gravity at the Earth's surface, and a sideways force of mv^2/r,owing to the centripetal acceleration of the volume, moving at velocity v around a central axis, a distance r from the center, both operate to shape the surface. A normal, reactive constraining force keeps this volume from moving elsewhere on the surface, and the slope of the surface must then be $\frac{dz}{dr} = \frac{v^2}{rg}$,when all is in equilibrium. Here, we align the z axis with the axis of rotation, and assume this is perfectly aligned with the vertical. Now, $v = r\omega, \omega$being the angular velocity with which we are spinning the dish containing the mercury. So the slope of the surface at some radial distance r out from the center must be

$$\frac{dz}{dr} = \frac{\omega^2}{g} r.$$

This integrates to

$$z = \frac{\omega^2}{2g} r^2.$$

where all but z, r are constants. Putting the equation into the Schwarzschild form:

$$-2Rz + r^2 = 0,$$

where

$$R = g/\omega^2,$$

and the conic constant $K = -1$,since no z^2 term appears. And so we have a parabolic surface of revolution. Some numbers here might be of interest: for a mirror 2.7 meters in diameter, and a focal ratio (focal length/diameter) of f:2, $R = 10.8$ meters. Putting 9.8 meters/second^{2}for g,$\omega = 0.9$ radian/second, or about 8.7 RPM. At the edge of the dish, the velocity would be 1.2 meters/sec.

Now, life isn't ever simple! Bora and Hickson had to overcome the obvious problem of rotating the mirror without vibrations ruining the smooth surface, and without air currents setting up little waves. And of course, the Earth is constantly rotating at 7.3×10^{-7}radians/second, which doesn't seem like much, but at a focal length of $R/2 = 5.4$ meters, this amounts to 0.39 mm/second. To track an object, the astronomers plan to shift pixels along in the detector. But the objects don't track in straight lines; they move in gently curved arcs, curved enough so that a corrector optic needed to be designed to distort the tracks into straight lines. Fortunately, this only has to be done for a given latitude. Coma for images at the edge of the field will also be a problem. Coma correctors are available, and are in common use, even in amateur telescopes.

A problem for larger diameters and higher speeds of rotation is possibly a Coriolis force distorting the mirror's shape. This is a pseudo force which results from the Earth's rotation, and it has magnitude $m\omega_e v \sin\phi$,where ϕ is the angle between the object's velocity and the Earth's axis; as you might suspect, it is a vector cross product. Let's say such a telescope is located on the Equator, and is rotating in a counter-clockwise direction. The outer north edge of the mirror is moving westward and will experience a downward acceleration. The outer south edge will experience an upward force; the east and west edges experience no force since they are moving parallel to the Earth's axis. For the above numbers this acceleration would be 8.7×10^{-6}meters/second^2, or about 1 part in a million compared to its weight. A part in a million at a meter radius is 1 micron, twice a wavelength of light. However, before getting excited over this, we should explore the nature of this aberration. First off, note that the aberration varies as $\sin\phi$,that is, the sine of the azimuth angle. This tells us that the mirror is distorted in having a tilted plane superimposed on it, which would result in a wavefront tilt. A tilt merely displaces the image; it doesn't distort it. And the effect is small: Converting a 1×10^{-6}radian to arc-seconds, we get a southward image shift of 0".2, hardly noticeable. This is the simplest case, but at other latitudes the effect is similar: a phase tilt.

VI. Planetary Motion and Kepler's Laws

Johannes Kepler, 1571 – 1630, discovered the elliptical motion of Mars, as a result of his collaboration with Tycho de Brahe, whose observations were the best of the day, and accurate enough for this purpose. He stated three empirical laws, which we will validate here: First, that the planets move in elliptical orbits with the Sun at a focus, second, that the planet's motion sweeps out equal areas in its orbital plane in equal times, and third that the square of its period is proportional to the cube of its mean distance from the Sun. We know him best for his contributions to optics and for these planetary discoveries, which later led to Newton's theory of planetary motion under gravitational forces. In an era of superstition and religious upheaval (Kepler was Protestant, and the Thirty Years War was being fought) it is a wonder he was able to accomplish this feat. His mother was imprisoned for witchcraft, but he successfully defended her and secured her release. His life was more turbulent than most, although he was not persecuted as badly as was Galileo. I think when Newton mentioned standing upon the shoulders of giants, he had Kepler in mind.

The following discussion is largely taken from Moulton, *An Introduction to Celestial Mechanics*, with some extra comments to help things along. Standing on the shoulders of Newton, we'll present the equations of motion:

$$\ddot{\mathbf{r}} = -\frac{k^2}{r^2}\mathbf{e_r}.$$

Because of its vector form, this is more than one equation of motion. As indicated by $\mathbf{e_r}$, the unit vector in the radial direction, the force acts radially. We actually phrase the equations here in terms of acceleration, the second time derivative of the radius vector (the two dots indicate the second time derivative - Newton's notation). k is a constant. The lion's share of the mass of the system resides in the Sun, and the coordinate r may be considered to originate at the center of the Sun. (This doesn't work for the more massive planets, but it is adequate for Mars. In the case of, say, Jupiter, r is then a difference in the position vectors originating at the center of mass of the system.) We'll work in polar coordinates, as this actually is the simplest way to approach the problem.

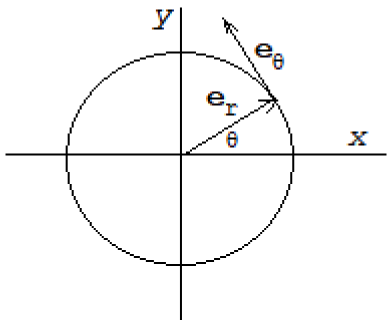

Fig. 11.

Taking the derivative of a vector in polar coordinates is tricky. We'll work out the derivatives for the polar unit vectors (in two dimensions) by expressing them in more familiar rectangular coordinates. See Fig. 11.

$$\mathbf{e_r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j},$$

$$\mathbf{e}_\theta = -\mathbf{sin}\theta\mathbf{i} + \cos\theta\mathbf{j},$$

where $\mathbf{i}, \mathbf{j}$ are unit vectors in the x, y directions, respectively. Now, we take the derivatives:

$$\frac{d\mathbf{e_r}}{dt} = (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt} = \dot{\theta}\mathbf{e}_\theta,$$

$$\frac{d\mathbf{e}_\theta}{dt} = (-\cos\theta\mathbf{i} - \sin\theta\mathbf{j})\frac{d\theta}{dt} = -\dot{\theta}\mathbf{e_r}.$$

Writing $\mathbf{r} = r\mathbf{e_r}$,we take its derivative twice:

$$\frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta,$$

$$\frac{d^2\mathbf{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\theta + r\ddot{\theta})\mathbf{e}_\theta.$$

So now this gives us two equations:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k^2}{r^2},$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

The reader can readily verify that the second equation has a solution:

$$r^2\dot{\theta} = h,$$

where h is a constant of the integration. A physics student will immediately recognize this as a statement of the conservation of angular momentum, mvr =const, only we have left the orbiting body's mass out of the equations. It also says that, no matter what the nature of the binding force is between Sun and planet, so long as it is central (has no θ component), angular momentum will be conserved. We'll return to this important result in a minute. But first, let's use this result in the preceding equation to eliminate the θ derivative:

$$\ddot{r} = \frac{h^2}{r^3} - \frac{k^2}{r^2}.$$

It is useful to make a substitution: $r = 1/u$,then

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$

$$\ddot{r} = -h\frac{d}{dt}\frac{du}{d\theta} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -u^2h^2\frac{d^2u}{d\theta^2},$$

$$\ddot{r} = \frac{h^2}{r^3} - \frac{k^2}{r^2} = -u^2h^2\frac{d^2u}{d\theta^2},$$

where we substituted for $\dot{\theta}$ wherever we could. Now, putting everything in terms of u, we have

$$k^2u^2 = u^3h^2 + u^2h^2\frac{d^2u}{d\theta^2} = u^2h^2\left(u + \frac{d^2u}{d\theta^2}\right),$$

$$\frac{d^2u}{d\theta^2} + u = \frac{k^2}{h^2}.$$

We recognize the homogeneous equation

$$\frac{d^2u}{d\theta^2} + u = 0,$$

as a wave equation in one dimension, and it has solution $C\cos(\theta - \theta_o)$,where A and θ_o are constants of the integration. In our case above, the general solution is found simply by adding $\frac{k^2}{h^2}$ to the homogeneous solution:

$$u = C\cos(\theta - \theta_o) + \frac{k^2}{h^2},$$

or putting things back in terms of r, we find we have an equation for a conic,

$$r = \frac{p}{1 + e\cos(\theta - \theta_o)},$$

$$p = \frac{h^2}{k^2},$$

$$e = C\frac{h^2}{k^2}.$$

If the orbiting planet has insufficient kinetic energy to escape the Sun, we have $e < 1$,and an elliptical orbit, thus satisfying Kepler's first law. Now we return to the result $r^2\dot{\theta} = h$. We note that in time dt the planet moves a distance

$rd\theta$ perpendicular to the radius vector, and the triangle thus formed has an area $dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}h dt$. We note that this area increment is only dependent upon time, thus fulfilling Kepler's second law that the radius vector sweep out equal areas in equal times. Finally, let's integrate our area for a full revolution of the planet around the sun, taking time T (the period). The area of the ellipse is $\pi ab = \frac{1}{2}hT$,and $b = \sqrt{1-e^2}a$, and for the ellipse, $p = a(1-e^2) = h^2/k^2$. Putting this all together:

$$A = \pi a^2 \sqrt{(1-e^2} = \frac{1}{2}k\sqrt{a(1-e^2)}T,$$

$$T^2 = \left(\frac{2\pi}{k}\right)^2 a^3,$$

confirming Kepler's third law.

VII. Summary

Conics have adorned mathematics since the time of Euclid and Apollonius, but in modern times, have shown their worth in practical applications. We have only discussed two: optics and celestial mechanics. There are of course others, for instance, the inertia ellipsoid used in describing problems of rotating rigid bodies. The shape of fast-rotating stars and planets is an oblate ellipsoid. And so on.

It all begins with a fictituous line, the directrix, and locus of a point which moves so as to keep a constant ratio of its distance between a focal point and that line. We saw how the directrix was related to a sliced cone, which of course is what gave these fascinating curves their name. We saw the application of surfaces of revolution of conics to optics; both reflecting systems and lens systems use them to advantage. We took a peek at the motion of planets, verifying Kepler's discoveries, while making use of Newton's mechanics.

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